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Restricted Chebyshev Centers of Bounded Subsets in Arbitrary Banach Spaces

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In this paper we consider the problem of the uniqueness of a restricted Chebyshev center of an arbitrary bounded subset F of an arbitrary Banach space B with respect to a subset $G \subsetneq B$. A key step is an explicit representation of the extreme points of the unit ball of the space dual to the space of affine mappings $f \mapsto \lambda \cdot f + b$ of F into B.

INTRODUCTION

Let X be a normed linear space, G and F subsets of X. Following [5], a solution g_0 of the problem

$$\inf_{g \in G} \sup_{f \in F} \| g - f \| = \sup_{f \in F} \| g_0 - f \| = r_G(F) \tag{1}$$

is called a restricted Chebyshev center of F with respect to G. The number $r_G(F)$ is the radius of F with respect to G. The set of such solutions is denoted $E_G(F)$.

Let F be a bounded subset of a Banach space B over a field $\mathbf{k}$ ($\mathbf{k} = \mathbf{R}$ or $\mathbf{k} = \mathbf{C}$) and let $\| \cdot \|_V$ be a monotone norm (i.e., $0 \leqslant \phi(f) \leqslant \psi(f)$ for every $f \in F$ implies $\| \phi \|_V \leqslant \| \psi \|_V$) defined on a subset V of[1] $C(F, \mathbf{R})$ which contains all functions of the form $\rho_{(\lambda, b)}(f) = \| \lambda \cdot f + b \|_B$ where $b \in B$, $f \in F$ and

[1] See the sentence following Theorem 2.

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$\lambda \in \mathbf{k}$. In particular, $\rho_{(1,-b)}(f) = \|f - b\|_B$. It is easy to verify (see [4]) that, for every monotone $\|\cdot\|_V$,

$$\| e_{(\lambda,b)} \|_{E(V)} \overset{\text{def.}}{=} \| \rho_{(\lambda,b)} \|_V, \tag{2}$$

too, is a norm in the vector space A of all affine mappings from F into B of the form $e_{(\lambda,b)}(f) = \lambda \cdot f + b$ where $b \in B, f \in F$ and $\lambda \in \mathbf{k}$.

Following [4], a solution g_0 of the problem

$$\inf_{g \in G} \| \rho_{(1,-g)} \|_V = \| \rho_{(1,-g_0)} \|_V = r_{G,V}(F) \tag{3}$$

is called a best $\|\cdot\|_{E(V)}$-simultaneous approximation of F by elements of G. It is clear that g_0 is a restricted Chebyshev center of F with respect to G if and only if g_0 is a best $\|\cdot\|_{E(V)}$-simultaneous approximation of elements of F by elements of G, where $\|\cdot\|_V$ is a supremum norm.

In [5, Theorem 5.1], Rosema and Smith proved a result concerning the uniqueness of restricted Chebyshev centers of F with respect to a convex subset G of $C(a, b)$, when F was a compact subset of $C(a, b)$. The uniqueness of a best $\|\cdot\|_{E(V)}$-simultaneous approximation of F by elements of convex subsets G of strictly convex Banach spaces B was established in [4, Theorem 4(b)] when F and $\|\cdot\|_V$ satisfy condition C.2, that is: For every $\phi_j = \rho_{(\lambda_j, b_j)}$, $j = 1, 2, 3$, with $\| \phi_1 \|_V = \| \phi_2 \|_V = \| \phi_3 \|_V$ and $\alpha \in \mathbf{R}$, $0 < \alpha < 1$, such that $0 \leqslant \phi_3(f) \leqslant \alpha \cdot \phi_1(f) + (1 - \alpha) \cdot \phi_3(f)$ for all $f \in F$, there exists $f' \in F$ satisfying $\phi_1(f') = \phi_2(f')$. In particular, in the case of $\|\cdot\|_V$ being a supremum norm, the class of bounded subsets $F \subset B$ satisfying condition C.2 includes all compact subsets of B (see *Example of Strict Monotonicity* in [4, Sect. 5]).

The case of an arbitrary bounded subset F of $C(a, b)$ was left open in [5]. For bounded subsets F satisfying condition C.2 Theorem 4(b) in [4] and Theorem 5.1 in [5] extend to the case of an arbitrary Banach space B, that is the following take place.

THEOREM 1. *Let G be a convex subset of a Banach space B. Assume that $\|\cdot\|_V$ is monotone, F is a bounded subset of B and condition C.2 holds. Suppose that a best approximation of any element $b \in F$ by elements of G is unique. Then the best $\|\cdot\|_{E(V)}$-simultaneous approximation of F by elements of G is unique.*

Proof. Consider $G_r =^{\text{def.}} \{b \in B$ such that $\| e_{(1,-b)} \|_{E(V)} = r\}$. Let $r = r_{G,V}(F)$. Then it is clear that G_r is convex. Assume b_1 and $b_2 \in G_r$, $b_1 \neq b_2$ (i.e. the contrary). Let $\phi_j(f) =^{\text{def.}} \|f - b_j\|_B$, where $b_3 = \frac{1}{2}(b_1 + b_2) \in G_r$ and $j = 1, 2, 3$. Then $\| \phi_j \|_V = r$, $j = 1, 2, 3$, and $0 \leqslant \phi_3(f) \leqslant \frac{1}{2}(\phi_1(f) + \phi_2(f))$ for every $f \in F$. Applying condition C.2 we find an element $f' \in F$ such that $\|f' - b_1\|_B = \|f' - b_2\|_B = \|f' - b_3\|_B$, where b_1, $b_2 \in G$ and f' is an element of F. The later contradicts our assumptions on G. Hence the cardinality of G_r is $\leqslant 1$, as required.

There are inherent difficulties in moving from the compact case to the case of an arbitrary bounded set F. Rozema and Smith's work rested upon the work of Laurent and Tuan [3], which in turn depended upon a result of Valadier [7] concerning an explicit representation of the subgradient of a convex functional. This expression was tractable whenever, F was a compact set. Whenever F was a bounded set, Valadier's result does not yield the easily applicable representation of the subgradient. Thus, this paper uses different techniques to extend the results of [4] and [5] on the uniqueness of a restricted Chebyshev center to the case of bounded sets in arbitrary Banach spaces.

In this paper the following notations or conventions will be used. For any normed space X, $U(X)$ denotes the closed unit ball. $S(X)$ denotes the boundary of $U(X)$. The dual space of X is denoted by X^*. The element θ represents the zero element of a generic vector space. The set $G^0 = \{\phi \in X^* / \phi(g) = 0$ for every $g \in G\}$ is the annihilator of G. The extreme points of a convex set K are denoted by ext K. The cardinality of a set F is denoted by $\#F$. We use the notation (λ, b) for the function $e_{(\lambda, b)}$.

Consider vector space A in the $\|\cdot\|_{E(V)}$-norm. For every $\mu \in \mathbf{R}$ and $v \in B^*$ let $(\mu, v) \in A^*$ be given by $(\mu, v)[(\lambda, b)] = \mu \cdot \lambda + v(b)$ for every $(\lambda, b) \in A$. A key step in the proof of the uniqueness of a restricted Chebyshev center for arbitrary bounded sets F and Banach spaces B is of interest in its own right. It is:

THEOREM 2. *Let F be a bounded subset of a Banach space B and $\|\cdot\|_V$ be a supremum norm. If $(\mu, v) \in$ ext $U(A^*)$, then $v \in$ ext $U(B^*)$.*

Let $C(F, B)$ be the space of bounded continuous functions from a bounded subset F of a Banach space B into B.

Remark 1. The extreme points of $U(C(F, B)^*)$ have a nice representation due to Singer [6, p. 197]. Since the use of the extreme points of the dual unit ball is of paramount importance in the representation of solutions of best approximation from finite-dimensional sets, we will appreciate the value of Singer's representation. In particular, if Φ is an element of ext $U(C(F, B)^*)$ and $T \in C(F, B)$, then Φ can be associated with a pair (f_0, Φ_0) in $F \times$ ext $U(B^*)$ via $\Phi(T) = \Phi_0(Tf_0)$. The difficulties in moving from the compact case to the case of bounded sets F lie in finding a useful representation for the extreme points of the dual of $C(F, B)$. Fortunately, we are able to find a representation of the extreme points of $U(A^*)$. The significance of Theorem 2 lies in the fact that there is no known representation of the extreme points of the dual ball of $C(F, B)$.

In Section 1 we recall the pertinent facts from [4] about the Banach space A and prove Theorem 2. Section 2 contains results on the uniqueness of a restricted Chebyshev center.

1. The Space A

We will use notation $\|\cdot\|$ for $\|\cdot\|_{E(V)}$, when $\|\cdot\|_V$ is a supremum norm, i.e. $\|T\| = \sup_{f\in F}\|T(f)\|_B$ for every $T \in C(F, B)$. If one lets G be any set in B and considers the solution of $\inf_{g\in G}\|(1, \theta) - (1, g)\| = \inf_{g\in G}\sup_{f\in F}\|f - g\|_B$, one sees that a best approximation to $e_{(1,0)}$ from the set $G(A) = \{e_{(0,g)} \mid g \in G\}$ is a restricted Chebyshev center of F with respect to G. Thus one can apply the theorems of best approximation of a point from a given set to the problems of restricted Chebyshev centers. By an abuse of notation we will continue to use G instead of $G(A)$ since the meaning is evident.

Proof of Theorem 2. Recall [4, Lemma 2.1] that A is a complete topological space and that A^* is identified with $R \oplus B^*$ by the formula $(\mu, v)[(\lambda, b)] = \mu \cdot \lambda + v(b)$ whenever F is a bounded subset of B with $\#F > 1$. To distinguish sets $U(A)$ and $U(A^*)$ for different sets F we use the following notation $U_F(A) = \{(\lambda, b) \in A / \|(\lambda, b)\| \leqslant 1\}$

and

$$U_F(A^*) = \left\{(\mu, v) \in A^* \text{ such that } \|(\mu, v)\|_{A^*} = \sup_{(\lambda,b)\in A} \frac{|\mu \cdot \lambda + v(b)|}{\|(\lambda, b)\|} \leqslant 1\right\}.$$

Consider $F_1 \subset F_2$, F_1 and F_2 bounded subsets of B. It is easy to see that $U_{F_1}(A) \supset U_{F_2}(A)$ and $U_{F_1}(A^*) \subset U_{F_2}(A^*)$. Moreover,

$$U_{F_0}(A) = \bigcap_{\substack{F\subseteq F_0\\ \#F<\infty}} U_F(A) \text{ and } U_{F_0}(A^*) = \bigcup_{\substack{F\subseteq F_0\\ \#F<\infty}} U_F(A^*). \tag{4}$$

Therefore

$$\text{ext } U_{F_0}(A^*) \subseteq \bigcup_{\substack{F\subseteq F_0\\ \#F<\infty}} \text{ext } U_F(A^*). \tag{5}$$

Assume now that $F \subset B$, $\#F < \infty$. Consider the natural imbedding Π: $A \to C(F, B)$, mappings Π^*: $C(F, B)^* \to A^*$ and Π^*: $U(C(F, B)^*) \to U_F(A^*)$ (the latter mapping is surjective by Hahn–Banach theorem). Then $\Pi^*(\text{ext } U(C(F, B)^*)) \supset \text{ext } U_F(A^*)$. But for a finite set $F \subset B$, it is easy to see that $C(F, B)^* = C(F, B^*)$, where for $T \in C(F, B^*)$ and $\Phi \in C(F, B)$ we let $T(\Phi) =^{\text{def.}} \sum_{f\in F} T(f)[\Phi(f)]$. Then $\text{ext } U(C(F, B)^*) = \text{ext } U(C(F, B)) = F \times \text{ext } U(B^*)$. Therefore for every $(\mu, v) \in \text{ext } U_F(A^*)$, there exists an element $(f, \phi) = T \in \text{ext } U(C(F, B^*))$ such that for all $(\lambda, b) \in A$ we have $\mu \cdot \lambda + v(b) = \phi(\lambda \cdot f + b) = \lambda \cdot \phi(f) + \phi(b)$. Then $\mu = \phi(f)$ and $v = \phi \in \text{ext } U(B^*)$, as required (in the case of a finite set F). For the general case it is sufficient to use (2). Since if $(\mu, v) \in \text{ext } U_{F_0}(A^*)$, then $(\mu, v) \in \text{ext } U_F(A^*)$ for some $F \mu F_0$ with $\#F < \infty$ and then $v \in \text{ext } U(B^*)$, as required. This completes the proof of the Theorem.

2. Restricted Chebyshev Centers of Bounded Sets

Following [1], an n dimensional subspace M of X is called an interpolating subspace if for each set of n linearly independent functionals $x_1^*, \ldots, x_n^*$ in $\operatorname{ext} U(X^*)$ and each set of scalars $c_1, \ldots, c_n$ there is a unique element $y \in M$ such that $x_i^*(y) = c_i$ for $i = 1, \ldots, n$. In this paper we use freely the following several equivalent definitions of interpolating subspaces from [1, Theorem 2.1]:

$M = \operatorname{span}(x_1, \ldots, x_n)$ is an interpolating subspace if and only if for each set $\{x_1^*, \ldots, x_n^*\} \subset \operatorname{ext} U(X^*)$ of linearly independent functionals $\det[x_i^*(x_j)] \neq 0$ holds, or equivalently if $y \in M$ and $x_i^*(y) = 0$ for $i = 1, \ldots, n$ then $y = 0$.

We now recover a result known essentially to Golomb [2] when F is a compact subset of $C(a, b)$.

Theorem 3. *Let B be a Banach space and F a bounded subset of B. Assume that G is an interpolating subspace of B of dimension n, disjoint with $E_G(F)$. Then $E_G(F)$ is a singleton.*

Proof. First of all $E_G(F)$ is nonvoid since the dimension of G is finite (see [4, Theorem 3]). Let $\tilde{G} = \operatorname{span}\{(1, \theta); G\} \subset A$. Then $\dim_{\mathbf{k}} \tilde{G} = n + 1 = \dim_{\mathbf{k}}(\tilde{G})^*$ and both mappings $p\colon A^* \to \tilde{G}^*$ and its restriction $p\colon U(A^*) \to U(\tilde{G}^*)$ are surjective by the Hanh–Banach theorem. Take $\gamma \in \tilde{G}^*$ such that $\gamma((0, g)) = 0$ for all $g \in G$ and $\gamma((1, \theta)) = r_G(F)$. It is easy to see that $\| \gamma \|_{\tilde{G}^*} = 1$ and $\gamma((1, \theta) - (0, g^*)) = r_G(F)$ for all $g^* \in E_G(F)$.

Let $\mathscr{F}(\tilde{G}^*) = \{\gamma \in U(\tilde{G}^*)$ such that $\gamma((1, \theta) - (0, g^*)) = r_G(F)$ for all $g^* \in E_G(F)\}$ and $\mathscr{F}(A^*) = p^{-1}(\mathscr{F}(\tilde{G}^*)) \subsetneq U(A^*)$. It is easy to see that $\mathscr{F}(\tilde{G}^*)$ is a face in $U(\tilde{G}^*)$, that is $\frac{1}{2}(\gamma' + \gamma'') \in \mathscr{F}(\tilde{G}^*)$ and $\gamma', \gamma'' \in U(\tilde{G}^*)$ implies that γ' and $\gamma'' \in \mathscr{F}(\tilde{G}^*)$. Therefore $\mathscr{F}(A^*) =^{\text{def.}} p^{-1}(\mathscr{F}(\tilde{G}^*)) \subsetneq U(A^*)$ is a face in $U(A^*)$. Thus $\operatorname{ext} \mathscr{F}(A^*) \subsetneq \operatorname{ext} U(A^*)$ and

$$\operatorname{ext} \mathscr{F}(\tilde{G}^*) \subsetneq p(\operatorname{ext} \mathscr{F}(A^*)) \subsetneq p(\operatorname{ext} U(A^*)). \tag{6}$$

Since $\mathscr{F}(\tilde{G}^*) \neq U(\tilde{G}^*)$, $\dim_{\mathbf{k}} \mathscr{F}(\tilde{G}^*) < \dim_{\mathbf{k}} U(\tilde{G}^*) = n + 1$ one may choose a collection of m ($m \leqslant n + 1$) linearly independent vectors $\gamma_j \in \operatorname{ext} \mathscr{F}(\tilde{G}^*)$, $1 \leqslant j \leqslant m$, such that $\gamma = \sum_{j=1}^m \alpha_j \cdot \gamma_j$, $\alpha_j \geqslant 0$ and $\sum_{j=1}^m \alpha_j = 1$. By (6), there is a collection of $T_j = (\mu_j, v_j) \in \operatorname{ext} U(A^*)$ (then $v_j \in \operatorname{ext} U(B^*)$ by Theorem 2), such that $p(T_j) = \gamma_j$. Let $(\gamma, v) = \sum_{j=1}^m \alpha_j(\mu_j, v_j)$. Then

$$0 = \gamma((0, g)) = v(g) = \sum_{j=1}^{m} \alpha_j \cdot v_j(g) \text{ for all } g \in G, \tag{7}$$

$$r_G(F) = \gamma_j((1, \theta) - (0, g^*)) = \mu_j - v_j(g^*) \qquad \text{for all} \quad g^* \in E_G(F). \tag{8}$$

Now, $v = \sum_{j=1}^{m} \alpha_j \cdot v_j \neq \theta$, since otherwise $r_G(F) = (\mu, \theta)[(1, -b)] = \mu \leqslant \|(\mu, \theta)\|_{A^*} \cdot \|(1, -b)\| \leqslant \|(1, -b)\|$ for all $b \in B$ and $r_G(F) = \|(1, -g^*)\|$ for a $g^* \in E_G(F)$, which contradicts $r_G(F) > r_B(F)$.

We claim that number q ($q \leqslant m$) of linearly independent v_j is equal $n + 1$. Otherwise we specify the first k that are linearly independent and write the remaining $m - k$ as linear combinations of the first k. Thus $\sum_{j=1}^{m} \alpha_j \cdot v_j = \sum_{j=1}^{k} \beta_j \cdot v_j \neq \theta$, with $\sum_{j=1}^{k} \beta_j \cdot v_j \in G^0$. Expand the set of v_j, $1 \leqslant j \leqslant k$, by including new $n - k$ elements v_j', $k + 1 \leqslant j \leqslant n$, from ext $U(B^*)$ such that the expanded set consists of linearly independent vectors. Set

$$\tau = \sum_{j=1}^{k} \beta_j \cdot v_j + \sum_{j=k+1}^{n} 0 \cdot v_j' = \sum_{j=1}^{n} \beta_j \cdot v_j' \neq \theta.$$

Clearly $\tau \in G^0$. If $G = \operatorname{span}\{x_j : j = 1,..., n\}$ then $\sum_{j=1}^{n} \beta_j \cdot v_j'(x_i) = 0$ for all $i = 1,..., n$. Since G is an interpolating subspace of B, we have $\det\{v_j'(x_i)\} \neq 0$ and all $\beta_j = 0$, which contradicts $\sum_{j=1}^{n} \beta_j \cdot v_j' = v \neq \theta$. Thus $q = m = n + 1$.

Now we may prove, using (8), that $E_G(F)$ is a singleton, since otherwise $v_j(g_1 - g_2) = 0$ for all $j = 1,..., n + 1$ and some $g_1, g_2 \in E_G(F)$, $g_1 \neq g_2$, which contradicts the fact that G is an interpolating subspace of B. This completes the proof of Theorem 3.

Let $x_1, ..., x_n$ be linearly independent elements of B and set

$$G = \left\{ \sum_{j=1}^{n} c_j \cdot x_j : a_j \leqslant c_j \leqslant b_j \right\}, \tag{9}$$

where, to avoid trivialities,

(i) a_j may be $+\infty$ but not $-\infty$,

(ii) b_j may be $-\infty$ but not $+\infty$,

(iii) $a_j \leqslant b_j$.

Set $I_1 = \{j: a_j = b_j\}$, $I_2 = \{j: a_j \neq b_j$, and not both $\pm\infty\}$, and $I_3 = \{1,..., n\} \backslash \{I_1 \cup I_2\}$.

The following result generalizes [5, Theorem 5.1] and [4, Theorem 4(b), the case of $\| \cdot \|_V = \| \cdot \|$sup] to the case of bounded subsets of an arbitrary Banach space B, and answers a question posed by the authors of [5]. We assume $\mathbf{k} = \mathbf{R}$.

THEOREM 4. *Let B be a Banach space and F a bounded subset of B. Let G be defined as in* (9) *and be disjoint with $E_B(F)$. Suppose that for every $J \subseteq I_2$,* span$\{x_j : j \in J \cup I_3\}$ *is an interpolating subspace of B. Then $E_G(F)$ is a singleton.*

Remark 2. In the case of a bounded set F satisfying condition C.2 (in particular for every compact set F) Theorems 3 and 4 are particular cases of Theorem 1.

Proof of Theorem 4. As in Theorem 3, $E_G(F)$ is nonvoid. We set $\tilde{G} =^{\text{def.}} \operatorname{span}\{(1, \theta); G\} \subseteq A$, and so $\dim_{\mathbf{R}} \tilde{G} = n + 1$. We want first to find a $\gamma \in \tilde{G}^*$ such that

$$\begin{gathered} \gamma((0, g)) \leqslant \gamma((0, g^*)) \text{ for all } g \in G \text{ and } g^* \in E_G(F), \\ \| \gamma \|_{\tilde{G}^*} = 1 \text{ and } \gamma((1, -g^*)) = -r_G(F) \text{ for all } g^* \in E_G(F). \end{gathered} \tag{10}$$

Let $S = \operatorname{span}\{-g_1 + E_G(F)\}$, where $g_1 \in E_G(F)$, and $\tilde{S} =^{\text{def.}} \operatorname{span}\{(1, -g_1); S(A)\} \subseteq \tilde{G}$. The convexity of $\| \cdot \|$ implies that $\inf_{g \in g_1 + S} \|(1, -g)\| = \|(1, -g^*)\| = r_G(F)$ for all $g^* \in E_G(F)$. Take $\tilde{\gamma} \in \tilde{S}^*$ defined by $\tilde{\gamma}((1, -g_1)) = -r_G(F)$ and $\tilde{\gamma}((0, g)) = 0$ for all $g \in S$. Then it is easy to verify that $\| \tilde{\gamma} \|_{\tilde{S}^*} = 1$. Note that $\tilde{\gamma}((1, -g)) = -r_G(F)$ for all $g \in E_G(F)$. Using the Hanh-Banach theorem we may find an extension $\gamma \in \tilde{G}^*$ of $\tilde{\gamma} \in \tilde{S}^*$ such that $\| \gamma \|_{\tilde{G}^*} = 1$. Then $| \gamma((1, -g))| \leqslant \|(1, -g)\| = r_G(F) = -\gamma((1, -g^*))$ for all $g \in G$ and $g^* \in E_G(F)$, which implies (10).

Let $\mathscr{F}(\tilde{G}^*) = \{\gamma \in U(G^*)$ such that $\gamma((1, \Theta) - (0, g^*)) = -r_G(F)$ for all $g^* \in E_G(F)\}$. Then $\gamma \in \mathscr{F}(\tilde{G}^*)$ and $\mathscr{F}(\tilde{G}^*)$ is a face of $U(\tilde{G}^*)$. Let $\gamma = \sum_{j=1}^{m} \alpha_j \cdot \gamma_j$, where $\gamma_j \in \operatorname{ext} U(\tilde{G}^*)$, $\alpha_j \geqslant 0$ and $\sum_{j=1}^{m} \alpha_j = 1$. Then $\gamma_j \in \operatorname{ext} \mathscr{F}(\tilde{G}^*)$ and $m = \dim_{\mathbf{R}} \mathscr{F}(\tilde{G}^*) + 1 \leqslant \dim_{\mathbf{R}} U(\tilde{G}^*) = n + 1$ (since $\mathscr{F}(\tilde{G}^*) \neq U(\tilde{G}^*)$). As in Theorem 3 the mapping $p: U(A^*) \to U(\tilde{G}^*)$ is surjective. Then there are $T_j = (\mu_j, v_j) \in \operatorname{ext} U(A^*)$ such that $p(T_j) = \gamma_j$, where $1 \leqslant j \leqslant m$ and $v_j \in \operatorname{ext} U(B^*)$. Let $(\mu, v) = \sum_{j=1}^{m} \alpha_j \cdot (\mu_j, v_j)$. It follows from (10), that

$$\max_{g \in G} v(g) = v(g^*) \text{ for all } g^* \in E_G(F). \tag{11}$$

Assume that Theorem 4 is not true. Then there are g_1 and $g_2 \in E_G(F)$, $g_1 \neq g_2$. Let $I = \{i$ such that the coefficients of x_i in g_1 and in g_2 in the decomposition (9) are different$\}$. Since v supports G, it supports the minimal face $\mathscr{F}$ of G containing g_1 and g_2. If $\# I = k$, then this face has 2^k extreme points.

We claim that $v(x_i) = 0$ for $i \leqslant I \cup I_3$. Using (11) the case of $i \in I_3$ is easy. If for some $i \in I$ $v(x_i) \neq 0$, we let $I^+ = \{i \in I: v(x_i) \geqslant 0\}$ and $I^- = \{i \in I: v(x_i) < 0\}$. Let $h_1 = \sum_{i \in I^+} b_i \cdot x_i + \sum_{i \in I^-} a_i \cdot x_i$ and $h_2 = \sum_{i \in I^+} a_i \cdot x_i + \sum_{i \in I^-} b_i \cdot x_i$ (h_1 and $h_2 \in \mathscr{F}$). Therefore $0 = v(h_1 - h_2) = \sum_{i \in I^+} (b_i - a_i) \cdot v(x_i) + \sum_{i \in I^-} (a_i - b_i) \cdot v(x_i) = \sum_{i \in I} d_i \cdot v(x_i)$ and sign $d_i =$ sign $v(x_i)$, whenever $v(x_i) \neq 0$. Thus $v(x_i) = 0$ for all $i \in I$, as we claimed.

Let $W = \operatorname{span}\{x_i: i \in I \cup I_3\}$. By the hypothesis W is an interpolating subspace of B; also $g_1 - g_2 \in W$.

As in Theorem 3 $v = \sum_{j=1}^{m} \alpha_j \cdot v_j \neq \Theta$, since otherwise, using (10), we obtain $r_G(F) = -(\mu, 0)[(1, -b)] = -\mu \leqslant \|(\mu, 0)\|_{A^*} \cdot \|(1, -b)\| = \|(1, -b)\|$ for all $b \in B$, which contradicts our assumption that $r_G(F) > r_B(F)$. We also claim that the number k of linearly independent v_i, $1 \leqslant i \leqslant m$, is greater or equal $q + 1 = \dim_{\mathbf{R}} W + 1$. Otherwise $(k < q + 1)$, specify the first k linearly independent and write the remaining $m - k$ as linear combination of the first k, so that $v = \sum_{j=1}^{m} \alpha_j \cdot v_j = \sum_{j=1}^{k} \beta_j \cdot v_j \neq \Theta$ and $\sum_{j=1}^{k} \beta_j \cdot v_j \in W^0$ (by the choice of W). Expand the set of v_j, $1 \leqslant j \leqslant k$, by including $q - k$ new elements v_j', $k + 1 \leqslant j \leqslant q$, from ext $U(B^*)$ such that the expanded set consists of linearly independent vectors. Set $\tau = \sum_{j=1}^{k} \beta_j \cdot v_j + \sum_{j=k+1}^{q} 0 \cdot v_j' = \sum_{j=1}^{q} \beta_j \cdot v_j' \neq \theta$. Then $\tau \in W^0$. If $W = \operatorname{span}\{e_1, ..., e_q\}$, then $\det\{v_j(e_i)\} \neq 0$, since W is an interpolating subspace of B. Therefore all $\beta_j = 0$, which contradicts $\sum_{j=1}^{q} \beta_j \cdot v_j' \neq \theta$. Hence $k \geqslant q + 1$, as it was claimed.

Now, using $p((\mu_j, v_j)) = \gamma_j \in \mathscr{F}(\tilde{G}^*)$ we have

$$-r_G(F) = \gamma_j((1, -g_i)) = \mu_j - v_j(g_i) \quad \text{for } i = 1, 2, 3 \quad \text{and} \quad j = 1, ..., m. \tag{12}$$

Therefore for k $(k \geqslant q + 1)$ linearly independent $v_j \in$ ext $U(B^*)$ $v_j(g_1 - g_2) = 0$ takes place. Since $g_1 - g_2 \in W$ is an interpolating subspace of B we obtain $g_1 = g_2$, which contradicts our choice of $g_1 \neq g_2$. This completes the proof of Theorem 4.

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